On the Partial Analytical Solution
of the Kirchhoff Equation

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Abstract. We derive a combined analytical and numerical scheme to
solve the (1+1)-dimensional differential Kirchhoff system. Here the ob-
ject is to obtain an accurate as well as an efficient solution process. Purely
numerical algorithms typically have the disadvantage that the quality of
the solutions decreases enormously with increasing temporal step sizes,
which results from the numerical stiffness of the underlying partial dif-
ferential equations. To prevent that, we apply a differential Thomas de-
composition and a Lie symmetry analysis to derive explicit analytical
solutions to specific parts of the Kirchhoff system. These solutions are
general and depend on arbitrary functions, which we set up according to
the numerical solution of the remaining parts. In contrast to a purely nu-
merical handling, this reduces the numerical solution space and prevents
the system from becoming unstable. The differential Kirchhoff equation
describes the dynamic equilibrium of one-dimensional continua, i.e. slen-
der structures like fibers. We evaluate the advantage of our method by
simulating a cilia carpet.

Key words: Differential Thomas Decomposition, Kirchhoff Rods, Lie
Symmetry Analysis, Partial Analytical Solutions, Partial Differential
Equations, Semi-analytical Integration.

1 Introduction

In general, to study algebraic properties of a polynomially nonlinear system of
partial differential equations (PDEs), the system has to be completed to involu-
tion [14]. In particular, by such a completion one can check the consistency of

the system, detect arbitrariness in its analytical solution, eliminate a subset of dependent variables, and verify whether some another PDE is a differential consequence of the system, i.e. verify whether this PDE vanishes on all solutions of the system. However, algorithmically, a polynomially nonlinear PDE system may not admit completion. Instead, one can decompose such a system into finitely many involutive subsystems with disjoint sets of solutions. This is done by the Thomas decomposition [2].[5]

In the present paper we apply first the differential Thomas decomposition to the equation system which comprises four linear PDEs in two independent and six dependent variables and also two quadratically nonlinear algebraic relations in the dependent variables. This partial differential-algebraic system is a parameter-free subsystem of the (1+1)-dimensional Kirchhoff equation. In doing so, we detect the arbitrariness in the general (analytical) solution to that subsystem. Then, by applying the classical theorem about conservative vector fields [5] and performing integration by parts we reduce the parameter-free subsystem to a system of three nonlinear PDEs in two independent variables that contains one arbitrary function.

As the next step, we perform the computer algebra based Lie symmetry analysis to the reduced system and discover that this system is equivalent (modulo Lie group transformations) to another system of three nonlinear PDEs in two independent variables (x, y) and two dependent variables f and g. The last system comprises two equations of zero Gaussian curvature for the surfaces $z = f(x, y)$ and $z = g(x, y)$ and one more equation that relates f and g. To construct the solution of the system we use the classical parametrization of a zero curvature surface [8]. As a result, we obtain a closed-form analytical solution to the parameter-free subsystem under consideration and prove that the solution is general. This opens up a wide range for applications of symbolic-numeric methods to the (1+1)-dimensional Kirchhoff equation. All symbolic computations related to the construction of the general analytical solution were done with Maple.

On that basis, we derive a combined analytical and numerical scheme to solve the (1+1)-dimensional Kirchhoff equation. This system of partial differential and algebraic equations describes the dynamic equilibrium of one-dimensional continua like slender structures, artificial fibers, human hair, cilia and flagella. Since their dynamical behavior act on different time-scales, numerical stiffness enters the problem and leads to huge instabilities in the case of basic purely numerical solution schemes. To prevent that, we exploit the explicit analytical expressions found for the solution to the parameter-free subsystem. In so doing, we set up the arbitrary functions in the exact solution according to the numerical solution of the Kirchhoff equation. In contrast to a purely numerical handling, this reduces the numerical solution space and prevents the Kirchhoff system from becoming unstable. The advantage is demonstrated by simulating a cilia carpet.

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[5] For a more-general overview of the formal algorithmic elimination for PDEs, we additionally refer to [13].

2 Kirchhoff's Rod Theory

For a detailed derivation of the special Cosserat theory of rods, we refer to our last year's contribution [12]. According to the formulation presented in there, the rods in classical Cosserat theory can undergo shear and extension. We can avoid this by setting the linear strains to $\boldsymbol{\nu} := (\nu_1, \nu_2, \nu_3) = (0, 0, 1)$ in local coordinates. Geometrically this means, that the angle between the director $\boldsymbol{d}_3$ and the tangent to the centerline, $\partial_s \boldsymbol{r}$, always remains zero (no shear) and that the tangent to the centerline always has unit length (no elongation).[6] We further reduce the system to two dimensions by setting $\kappa_3 = \omega_3 = \upsilon_3 = 0$. Finally we separate the resulting equations into two PDE subsystems

$$\rho A \partial_t \boldsymbol{v} = \partial_s \boldsymbol{n} + \boldsymbol{f}, \tag{1}$$

$$\rho I \partial_t \boldsymbol{\omega} = \partial_s \boldsymbol{m} + \operatorname{adiag}(1, -1)\, \boldsymbol{n} + \boldsymbol{l} \tag{2}$$

and

$$\partial_t \boldsymbol{\kappa} = \partial_s \boldsymbol{\omega}, \tag{3}$$

$$\partial_s \boldsymbol{v} = \operatorname{adiag}(1, -1)\, \boldsymbol{\omega}, \tag{4}$$

and additionally a set of constraints

$$\boldsymbol{\omega} \operatorname{adiag}(1, -1)\, \boldsymbol{\kappa}^{\mathsf{T}} = 0, \tag{5}$$

$$\boldsymbol{v} \operatorname{adiag}(1, -1)\, \boldsymbol{\kappa}^{\mathsf{T}} = 0 \tag{6}$$

with an antidiagonal matrix

$$\operatorname{adiag}(1, -1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Please note, that the equations (3)–(4) are parameter free, whereas equations (1)–(2) include parameters ρ, A, I.

3 Derivation of Explicit Analytical Solution

In this section, we derive an exact analytical solution to (3)–(4) under the constraints (5)–(6).

3.1 Arbitrariness in General Solution

To determine the functional arbitrariness in the general analytical solution to equations (3)–(6) with non-vanishing values of the dependent variables, we apply

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[6] For a more-general overview of the special Cosserat and Kirchhoff rod theory, we refer to [1].

to them the differential Thomas decomposition [2,13] by using its implementation in Maple[7] under the choice of a degree-reverse-lexicographical ranking with

$$s \succ t; \qquad \omega_2 \succ \omega_1 \succ \kappa_2 \succ \kappa_1 \succ \upsilon_2 \succ \upsilon_1 \,.$$

Then, under the condition $\omega_1\omega_2\kappa_1\kappa_2\upsilon_1\upsilon_2 \neq 0$, the decomposition algorithm outputs two involutive differential systems. One of them is given by the set of equations

$$\upsilon_1\underline{\omega_2} - \upsilon_2\omega_1 = 0\,,$$
$$\underline{\partial_s\omega_1} - \partial_t\kappa_1 = 0\,,$$
$$\upsilon_1\underline{\kappa_2} - \upsilon_2\kappa_1 = 0\,,$$
$$\upsilon_1^2\omega_1^2\underline{\partial_s\kappa_1} - \kappa_1^2\upsilon_1\omega_1\partial_t\upsilon_1 - 2\kappa_1\upsilon_1^2\omega_1\partial_t\kappa_1 + \kappa_1\upsilon_2\omega_1^3 = 0\,,$$
$$\underline{\partial_s\upsilon_2} + \omega_1 = 0\,,$$
$$\kappa_1\upsilon_1^2\underline{\partial_t\upsilon_2} - \kappa_1\upsilon_2\upsilon_1\partial_t\upsilon_1 + (\upsilon_2^2 + \upsilon_1^2)\omega_1^2 = 0\,,$$
$$\upsilon_1\underline{\partial_s\upsilon_1} - \upsilon_2\omega_1 = 0$$

and inequations

$$\upsilon_1 \neq 0\,,\ \upsilon_2 \neq 0\,,\ \kappa_1 \neq 0\,,\ \omega_2 \neq 0$$

with underlined *leaders*, i.e., the highest ranking derivatives occurring in the equations. The second involutive system in the output contains the equation $\upsilon_1^2 + \upsilon_2^2 = 0$ which has no real solution under our assumption of non-vanishing dependent variables.

The differential dimensional polynomial [11] computed for the left-hand sides of the above equation system

$$3\binom{l+1}{l} + 1$$

shows that the general analytical solution depends on three arbitrary functions of one variable.

Remark 1. This functional arbitrariness can also be understood via well-posedness of the Cauchy (initial value) problem [7,14]. For the above involutive system, the Cauchy problem has a unique analytical solution in a vicinity of a given initial point (s_0, t_0) if $\upsilon_2(s_0, t_0)$ is an arbitrary constant and $\kappa_1(s_0, t), \omega_1(s_0, t), \upsilon_1(s_0, t)$ are arbitrary functions of t, analytical at the point $t = t_0$.

Remark 2. In the below described procedure, to construct a solution to (3)–(6) we have to solve a number of intermediate partial differential equations. In doing so, it suffices to take into account the functional arbitrariness in the solutions to those equations and to neglect the arbitrariness caused by arbitrary constants.

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[7] The Maple package DIFFERENTIALTHOMAS is freely available and can be downloaded via http://wwwb.math.rwth-aachen.de/thomasdecomposition/.

3.2   Reduction to Two Dependent Variables

If D is an open simply connected subset of the real space $\mathbb{R}^2$ of the independent variables (s,t), then the partial differential equations (3) define conservative vector fields (κ_1, ω_1) and (κ_2, ω_2) on D.

By the classical theorem of calculus [5][8] there exist functions $p_1(s,t)$ and $p_2(s,t)$ such that

$$\kappa_1 = \partial_s p_1 \,, \quad \omega_1 = \partial_t p_1 \,, \quad \kappa_2 = \partial_s p_2 \,, \quad \omega_2 = \partial_t p_2 \,. \tag{7}$$

If one takes into account these relations, then the application of the same theorem to equation (4) transforms them into the equivalent form

$$\upsilon_1 = \partial_t f \,, \quad \upsilon_2 = \partial_t g \,, \quad p_1 = -\partial_s g \,, \quad p_2 = \partial_s f \,, \tag{8}$$

where $f = f(s,t)$ and $g = g(s,t)$ are some functions.

Therefore, the parameter-free system (3)–(6) is equivalent to (i.e. it has the same solution space as) the following system of two nonlinear PDEs:

$$f_{st}g_{ss} - g_{st}f_{ss} = 0 \,, \tag{9}$$
$$g_{ss}g_t + f_{ss}f_t = 0 \,. \tag{10}$$

Here and below we use the following notation: $f_s \equiv \partial_s f$, $f_t \equiv \partial_t f$, $f_{st} \equiv \partial_s \partial_t f$, et cetera. We look for a solution of (3)–(6) such that

$$\omega_1 \omega_2 \kappa_1 \kappa_2 \upsilon_1 \upsilon_2 \neq 0 \,. \tag{11}$$

The equations (9)–(10) form a linear system in f_{ss} and g_{ss}. It has non-trivial solutions only if

$$g_{st}g_t + f_{st}f_t = 0 \tag{12}$$

what is equivalent to the equality $\omega_1 \upsilon_2 - \omega_2 \upsilon_1 = 0$ that follows from (5)–(6).

Obviously, equation (12) admits an integration by parts which yields

$$(g_t)^2 + (f_t)^2 - h(t) = 0 \,,$$

where $h(t)$ is an arbitrary function. Therefore, the system (3)–(6) is equivalent to the system

$$g_{ss}g_t + f_{ss}f_t = 0 \,, \tag{13}$$
$$(g_t)^2 + (f_t)^2 - h = 0 \,, \quad h_s = 0 \,. \tag{14}$$

Now we apply the Thomas decomposition algorithm in its implementation in Maple [2] for the differential elimination of the function f from the system (13)–(14) and independently the differential elimination of the function g from the same system. In doing so, inequation (11) expressed in terms of f and g is taken into account. As a result of each these two eliminations, equation (13) is replaced with another equation whereas equations (14) are preserved by the elimination.

Consider now the PDE system which comprises two equations in (14) and two more equations[9] obtained by the differential elimination of the functions f

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[8] The proof is also given in `http://www.owlnet.rice.edu/~fjones/chap12.pdf`.
[9] Note that these equations are symmetric under the interchange of f with g.

and g as explained above:

$$g_{ss}g_t h_t + 2\,h \cdot (g_{st})^2 - 2\,h\,g_{ss}g_{tt} = 0\,, \tag{15}$$

$$f_{ss}f_t h_t + 2\,h \cdot (f_{st})^2 - 2\,h\,f_{ss}f_{tt} = 0\,, \tag{16}$$

$$(g_t)^2 + (f_t)^2 - h = 0\,, \quad h_s = 0\,. \tag{17}$$

Remark 3. If one fixes the ranking and applies the differential Thomas decomposition to the equation systems (13)–(14) and (15)–(17) extended with inequation (11), then in both cases the same set of equations and inequations is output. This shows explicitly the equivalence of both PDE systems as having the same solution space.

3.3   Lie Symmetry Analysis

Since the PDE system (15)–(17) is symmetric with respect to f and g, it suffices to apply the methods of the Lie symmetry analysis [9] to (16)–(17). As in our paper [12], to perform related symbolic computations we use the Maple package DESOLVE [4] and its routine *gendef* to generate the determining linear PDE system for the infinitesimal point transformations. Then we apply the Maple package JANET[10] [3] to complete the determining system to involution[11].

As a result, the involutive determining system containing twenty-three one-term and two-term linear PDEs is easily solvable by the Maple routine *pdsolve*. This gives for the coefficients of the infinitesimal Lie symmetry generator

$$\mathcal{X} = \xi_1(s,t,f,h)\,\partial_s + \xi_2(s,t,f,h)\,\partial_t + \eta_1(s,t,f,h)\,\partial_f + \eta_2(s,t,f,h)\,\partial_h \tag{18}$$

the following expressions:

$$\eta_1 = c_1 s + c_2 f + c_3\,, \ \ \eta_2 = -\frac{\mathrm{d}\phi}{\mathrm{d}t} + c_4\,, \ \ \xi_1 = c_5 s + c_6 f + c_7\,, \ \ \xi_2 = \phi\,,$$

where c_i $(1 \le i \le 7)$ are arbitrary constants, and $\phi = \phi(t)$ is an arbitrary function.

As we are looking for the functional arbitrariness only (see Remark 2), we put $c_i = 0$. The Lie symmetry group defined by the generator (18) is obtained by solving the Lie equations

$$\frac{ds}{d\alpha} = \xi_1\,, \quad \frac{dt}{d\alpha} = \xi_2\,, \quad \frac{df}{d\alpha} = \eta_1\,, \quad \frac{dh}{d\alpha} = \eta_2\,.$$

For the independent variables this gives the one-parameter transformation group

$$\tilde{s} = s\,, \quad \tilde{t} = \Phi(t,\alpha)\,.$$

Here α is a group parameter and Φ is an arbitrary function such that $\phi(t) = \partial_\alpha \Phi|_{\alpha=0}$. Now we can formulate and prove one of the main theoretical results of the present paper.

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[10] The Maple package JANET is freely available and can be downloaded via `http://wwwb.math.rwth-aachen.de/Janet/`.

[11] The same output can also be generated using the package DIFFERENTIALTHOMAS.

Lemma 1. *Let $h(t) > 0$ be a given function. Then for any solution to the equation (16) there exists an invertible transformation of variables $(s, t) \rightarrow (x, y)$ of the form*

$$x = s, \quad y = \Psi(t) \tag{19}$$

such that the solution is an image of a solution to the equation

$$f_{xx} f_{yy} - (f_{xy})^2 = 0 . \tag{20}$$

Proof. First, we note that the transformation (19) is invertible if and only if $\Psi_t \neq 0$. Second, we apply this transformation to (20). This gives

$$\frac{f_{ss} f_{tt} - (f_{st})^2}{f_{ss} f_t} = \frac{\Psi_{tt}}{\Psi_t} . \tag{21}$$

On the other hand, equation (16) can be rewritten as

$$\frac{f_{ss} f_{tt} - (f_{st})^2}{f_{ss} f_t} = \frac{h_t}{2h} . \tag{22}$$

We see that at $(\Psi_t)^2 = h$ the right-hand sides of (21) and (22) coincide.   □

This lemma immediately implies the following proposition.

Proposition 1. *The PDE system (15)–(17) can be obtained from the system*

$$f_{xx} f_{yy} - (f_{xy})^2 = 0 , \tag{23}$$

$$g_{xx} g_{yy} - (g_{xy})^2 = 0 , \tag{24}$$

$$(f_y)^2 + (g_y)^2 - 1 = 0 \tag{25}$$

by the transformation (19) of the independent variables with $h = (\Psi_t)^2$.

Thus, we reduced the parameter-free subsystem (3)–(6) of the Kirchhoff equation to the equivalent system (23)–(25). In doing so, the independent variables of the latter system are related to the independent variables of the former one by the equalities (19). The advantages of the reduced system are:

1. there are only two dependent variables;
2. the system is symmetric with respect to the dependent variables;
3. they are separated in (23) and (24);
4. the differential equation of the form (23) is very well known in the theory of analytical surfaces and deeply studied in the literature [10]. It means that the surface $z = f(x, y)$ has zero Gaussian curvature. Such surfaces are locally isometric to a plane and admit a parametrization constructed in the classical paper [8]. In the following subsection we use this parametrization to derive the general analytical solution to (3)–(6).

3.4 Parametrized Analytical Solution

In [8] a parametric form of a general analytical solution to an isolated zero curvature equation was found. For equation (23) this parametrization reads

$$f = a_1(u)v + b_1(u) \,, \quad x = a_2(u)v + b_2(u) \,, \quad y = a_3(u)v + b_3(u) \,. \qquad (26)$$

Here a_i and b_i are analytical functions of the parameter u.

Remark 4. In general, the function g may have the parameterizations of x and y such that $a_i(u), b_i(u)$ $(i = 2, 3)$ are different from that in (26). We assume, however, that both functions f and g have the same parametrizations of x and y so that $a_2 \neq 0$.

Our assumption allows to express the parameter v linearly in terms of x and leads to a rather compact parametrization of the functions f and g of the form

$$f = A_1(u)\, x + B_1(u) \,, \quad y = A(u)\, x + B(u) \,, \quad g = M(u)\, x + N(u)$$

with six functions. However, these functions are not arbitrary. By taking the first and second order derivatives of both sides in the equality

$$f\left(x, A(u)\, x + B(u)\right) = A_1(u)x + B_1(u)$$

with respect to x and u we obtain a linear equation system in $f_x, f_y, f_{xx}, f_{yy}, f_{xy}$. Its solution allows to express the partial derivatives f_{xx}, f_{yy}, f_{xy} in terms of the functions $A(u), A_1(u), B(u), B_1(u)$ and their first derivatives. Then, the substitution of these expressions into equation (23) shows that this equation is satisfied if

$$(A_1)_u = P(u)\, A_u \,, \quad (B_1)_u = P(u)\, B_u \,, \qquad (27)$$

where $P(u)$ can be any function. Similarly, equation (24) is rewritten as two differential equalities

$$M_u = Q(u)\, A_u \,, \quad N_u = Q(u)\, B_u \qquad (28)$$

with unspecified $Q(u)$. Next, equation (25) implies the algebraic equality $Q(u)^2 + P(u)^2 = 1$ or equivalently

$$Q(u) = \sin(C(u)) \,, \qquad P(u) = \cos(C(u)) \qquad (29)$$

with arbitrary $C(u)$.

As the next step, we apply the following transformation of the independent variables, which are inverse to (19):

$$s = x \,, \quad t = F(y)$$

with $F = \Psi^{-1}$. For this purpose, we use the relations

$$f\left(s, F\left(A(u)\, s + B(u)\right)\right) = A_1(u)s + B_1(u) \,,$$
$$g\left(s, F\left(A(u)\, s + B(u)\right)\right) = M(u)s + N(u)$$

together with (27)–(29) to compute all partial derivatives of the functions f and g with respect to s and t that occur in (7) and (8).

This gives the analytical solution

$$
\begin{aligned}
\boldsymbol{\kappa} &= -\frac{A^2(u)C'(u)}{A'(u)s + B'(u)} \begin{pmatrix} \cos(C(u)) \\ \sin(C(u)) \end{pmatrix}, \\
\boldsymbol{\omega} &= \frac{A(u)C'(u)}{F'(A(u)s + B(u))(A'(u)s + B'(u)))} \begin{pmatrix} \cos(C(u)) \\ \sin(C(u)) \end{pmatrix}, \\
\boldsymbol{v} &= \frac{1}{F'(A(u)s + B(u))} \begin{pmatrix} \cos(C(u)) \\ \sin(C(u)) \end{pmatrix}, \quad t = F\left(A(u)\,s + B(u)\right).
\end{aligned}
\tag{30}
$$

Here $A(u), B(u), C(u)$ are arbitrary analytical functions of one variable, F is an arbitrary analytical function of the argument $A(u)\,s + B(u)$, and the prime mark denotes differentiation of a function with respect of its argument.

Substituting (30) in equations (3)–(6) shows that these equations are satisfied for any functions A, B, C, F. In Section 3.1 we saw that the general analytical solution depends on three arbitrary functions, each of one variable. The following theorem shows that the functional arbitrariness in (30) is superfluous.

Theorem 1. *Set $B(u) = u$ in (30). Then the analytical solution*

$$
\begin{aligned}
\boldsymbol{\kappa} &= -\frac{A^2(u)C'(u)}{A'(u)s + 1} \begin{pmatrix} \cos(C(u)) \\ \sin(C(u)) \end{pmatrix}, \\
\boldsymbol{\omega} &= \frac{A(u)C'(u)}{F'(A(u)s + u)(A'(u)s + 1))} \begin{pmatrix} \cos(C(u)) \\ \sin(C(u)) \end{pmatrix}, \\
\boldsymbol{v} &= \frac{1}{F'(A(u)s + u)} \begin{pmatrix} \cos(C(u)) \\ \sin(C(u)) \end{pmatrix}, \quad t = F\left(A(u)\,s + u\right).
\end{aligned}
\tag{31}
$$

to (3)–(6) is general.

Proof. Assume without loss of generality that the initial point is $(s_0, t_0) = (0,0)$ and pose the Cauchy problem for (3)–(6) in a vicinity of this point. Then the well-posed Cauchy problem reads (see Remark 1):

$$
v_1(0, t) = f_1(t), \quad \omega_1(0, t) = f_2(t), \quad \kappa_1(0, t) = f_3(t), \quad v_2(0, 0) = C_1, \tag{32}
$$

where f_1, f_2, f_3 are analytic functions and C_1 is an arbitrary constant. Substitution of (31) into (32) implies the equations

$$
\begin{aligned}
\frac{\cos(C(u))}{F'(u)} &= f_1(F(u)), \\
\frac{C'(u)A(u)\cos(C(u))}{F'(u)} &= f_2(F(u)), \\
-C'(u)A^2(u)\cos(C(u)) &= f_3(F(u)), \\
\frac{\sin(C(0))}{F'(0)} &= C_1, \\
F(0) &= 0.
\end{aligned}
\tag{33}
$$

Now we aim to show that for any predetermined functions satisfying

$$f_1(t) \neq 0, \quad f_2(t) \neq 0, \quad f_3(t) \neq 0, \quad v_2(0,0) = C_1 \neq 0$$

it is always possible to match up functions $C(u), A(u), F(u)$ such that in a vicinity of $(0,0)$ the solution (31) satisfy the initial data (32). The squared second equation in (33) divided by the third equation gives the equality

$$-\frac{C'(u)\cos(C(u))}{(F'(u))^2} = \frac{f_2^2(F(u))}{f_3(F(u))}$$

which together with the first and third equations in (33) implies

$$C'(u) = -\frac{f_2^2(F(u))\cos(C(u))}{f_3(F(u))f_1^2(F(u))}, \quad C(0) = \arctan\left(\frac{C_1}{f_1(0)}\right),$$

$$F'(u) = \frac{\cos(C(u))}{f_1(F(u))}, \quad F(0) = 0, \tag{34}$$

$$A(u) = -\frac{f_3(F(u))f_1(F(u))}{f_2(F(u))\cos(C(u))}.$$

If $f_1(0) \neq 0$, $f_2(0) \neq 0$, $f_3(0) \neq 0$, then in some vicinity of zero the problem (34) has the unique solution. Indeed, the first two equations in (34) define a Cauchy problem with regular right-hand sides, and the last equation sets up the function $A(u)$ explicitly. In doing so, $C(0) \in (\pi/2, -\pi/2)$ and, hence, $\cos(C(0)) \neq 0$. In a vicinity of $u = 0$ all factors occurring in $C'(u), F'(u), A(u)$ do not vanish, and the unique solution of this system satisfies (33) identically. The last statement is easily verified by substituting (34) in (33). $\qquad\square$

4   Numerical Experiments

The resulting analytical solutions (31) for the system (3)–(4) under the constraints (5)–(6) contain four parameterization functions, which can be determined by the numerical integration of the equations (1)–(2). The substitution of the equations (31) into the equations (1)–(2), the replacement of the spatial derivatives with central differences, and the replacement of the temporal derivatives according to the numerical scheme of a forward Euler integrator, leads to an explicit expression.[12] Iterating over this recurrence equation allows for the simulation of the dynamics of a Kirchhoff rod.

We present this in the context of the simulation of the beating pattern of a cilium. Such kind of scenarios are of interest in the context of simulations in biology and biophysics, since this kind of patterns occur widely in nature. For example cilia carpets in the interior of the lung are responsible for the mucus transport, see [6]. The results are shown in Figure 1. Compared to a purely numerical handling using a discretization of the equations (1)–(4) in a similar

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[12] We do not explicitly write out the resulting equations here for brevity.

way, the step size can be increased by three orders of magnitude, which leads to an acceleration of two orders of magnitude. This is of special interest for complex systems like cilia carpets, in which multiple cilia are beating in parallel with phase differences in order to produce the appearance of a wave. This is illustrated in Figure 1.
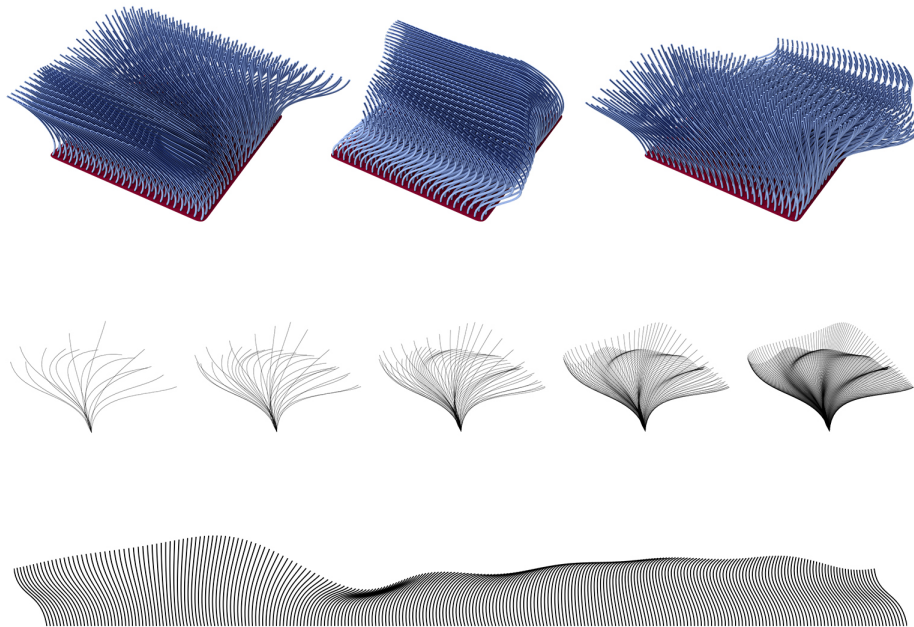

Fig. 1: Simulation of a cilia carpet (top) composed of multiple cilia beating in a metachronal rhythm (middle). This produces the appearance of a wave (below).

5   Conclusion

In this contribution, we have studied the two-dimensional Kirchhoff system. To prevent from numerical problems which result from the numerical stiffness of the underlying differential equations, we developed a combined, partially symbolic and partially numeric, integration scheme. Within the symbolic part we constructed an explicit analytical solution to the parameter-free part of the Kirchhoff system and proved that this solution is general. The application of the analytical solution prevents from numerical instabilities and allows for highly accurate and efficient simulations. This was demonstrated for the scenario of a cilia carpet, which could be efficiently simulated with an acceleration of two orders of magnitude compared to a purely numerical handling.

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